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ON THE PRINCIPLE OF INVARIANCE IN THE THEORY OF INHOMOGENEOUSLY AGEING SHELLS*

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It is proved that a general governing relationship results for an inhomogeneously ageing hereditary medium under fairly broad assumptions from the principle of material independence from the reference system /1/, also later called the principle of invariance.

It has been established experimentally that for a broad class of structural materials /2, 3/ (concrete, polymers, metals, etc.) the age of the material, measured from the time of its transition into the solid state, is the same substantial chracteristic governing its physicomechanical properties as the temperature, the humidity of the environment, etc. The instability of the properties that takes place under constant external conditions is due to the physocochemical processes occurring in the material whose mechanism has still been studied insufficiently. Such an instability, called ageing, results in the fact that the age of the material enters into the defining equation in addition to the remaining quantities.

The theory of the creep of ageing materials was developed in /2/ for concrete. The defining equations of inhomogeneously ageing materials possessing an age inhomogeneity were first obtained in /4, 5/. However, in the mechanics courses translated into Russian, ageing media are excluded from consideration on the basis of the assertion deduced from the principle of invariance that the absolute time t cannot enter the defining equation as a parameter (/6/ p.122, /7/, p.229). In this connection the question arises of the admissibility, in principle, of a phenomenological consideration of ageing media. As is shown later, the principle of material independence from the reference system not only does not prevent such a consideration, but also enables a general defining equation to be obtained for an inhomogeneously ageing medium.

We restrict the subsequent examination to so-called simple media whose defining equation has the form /1/

$$\sigma(t, \mathbf{X}) = \Phi(t, \mathbf{F}^t(\mathbf{X}), \mathbf{X}) \tag{1}$$

Here $\mathbf{X}$ is the site of a material point in a fixed reference configuration $\varkappa$, and $\mathbf{x} = \chi(t, \mathbf{X})$ is the site of this point in the actual configuration at the time t, σ is the Cauchy stress tensor, $\mathbf{F}(t, \mathbf{X}) = \nabla_{\mathbf{X}}\chi(t, \mathbf{X})$ is the gradient of the mapping $\chi(t, \mathbf{X})$ of the reference into the actual configuration, called the strain gradient, and $\mathbf{F}^t(\mathbf{X})$ denotes prehistory of the function $\mathbf{F}$ up to the time t and defined as follows /1/:

$$\mathbf{F}^t(\mathbf{X}) = (\mathbf{F}(t-s, \mathbf{X}), s \geqslant 0) \tag{2}$$

Finally, Φ is a functional (in the second argument) dependent parametrically on t and $\mathbf{X}$. Since the reference configuration is henceforth fixed, the dependence of Φ on $\varkappa$ is not indicated explicitly. We treat the second rank tensors as linear mappings as in /1/.

All possible changes of the reference system generate a group of space-time transformations

$$\mathbf{x}^* = \mathbf{x}_0^*(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{x}_0(t)) \quad t^* = t + a \tag{3}$$

where $\mathbf{x}_0^*, \mathbf{x}_0$ are arbitrary vector functions of time, $\mathbf{Q}(t)$ is an arbitrary orthogonal tensor, and a is an arbitrary constant. Here the first formula yields the group of solid body motions, and the second the shift of the time origin. The transformations of the tensors σ and $\mathbf{F}$ corresponding to relationships (3) will be

$$\sigma^*(t^*, \mathbf{X}) = \mathbf{Q}(t)\sigma(t, \mathbf{X})\mathbf{Q}^T(t), \quad \mathbf{F}^*(t^*, \mathbf{X}) = \mathbf{Q}(t)\mathbf{F}(t, \mathbf{X}) \tag{4}$$

where the superscript T denotes transposition.

The principle of invariance requires that for an arbitrary dynamic process (χ, σ) satisfying (1), all possible processes (χ^*, σ^*) obtained from (χ, σ) by the transformation (3), (4), i.e.,

$$\sigma^*(t^*, \mathbf{X}) = \Phi(t^*, \mathbf{F}^{*t^*}(\mathbf{X}), \mathbf{X}) \tag{5}$$

would also satisfy this Eq.(1).

The requirement considered restricts the class of functionals Φ governing the material properties. Assuming the known constraints resulting from (3)-(5) to be satisfied for $a = 0$,

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i.e., from invariance relative to space transformations /1/, by following /6, 7/, we prove that the functional Φ should be independent of t for the defining Eq.(1). Assuming $\mathbf{Q} = \mathbf{I}$ (the identity transformation), $\mathbf{x}_0^* = \mathbf{x}_0 = 0$, $a \neq 0$ in (2)-(4), we will have

$$\sigma^*(t^*, \mathbf{X}) = \sigma(t, \mathbf{X}), \quad \mathbf{F}^*(t^*, \mathbf{X}) = \mathbf{F}(t, \mathbf{X}) \tag{6}$$
$$\mathbf{F}^{*t^*}(\mathbf{X}) = (\mathbf{F}^*(t-s, \mathbf{X}), s \geqslant 0) = (\mathbf{F}(t-s, \mathbf{X}), s \geqslant 0) = \mathbf{F}^t(\mathbf{X})$$

Hence and from (1) and (5) we obtain the identity

$$\Phi(t, \mathbf{F}^t(\mathbf{X}), \mathbf{X}) = \Phi(t+a, \mathbf{F}^t(\mathbf{X}), \mathbf{X}) \tag{7}$$

Setting $a = -t$ here, we obtain the required relationship

$$\Phi(t, \mathbf{F}^t(\mathbf{X}), \mathbf{X}) = \Phi(0, \mathbf{F}^t(\mathbf{X}), \mathbf{X}) \tag{8}$$

In the authors' opinion, the correct interpretation of relationship (8) is not that the material reaction Φ is generally independent of the time but just that the time t cannot enter explicitly into the reaction Φ in any reference system in the simplest form (1).

We will consequently try to find the defining equation of an ageing material in a broader class of relationships. Assuming that we speak about a solid, we let $t_0(\mathbf{X})$ denote the time of formation (generation) of the solid material (from the liquid or gas phase) in the neighbourhood of the point $\mathbf{X}$. Being already a solid, the material can later undergo phase rearrangements at the times $t_1(\mathbf{X}), t_2(\mathbf{X}), \ldots t_N(\mathbf{X})$. Consequently, it is natural to seek the desired defining equation in the form

$$\sigma(t, \mathbf{X}) = \Phi_1(t, t_0, t_1, t_2, \ldots t_N, \mathbf{F}^t(\mathbf{X}), \mathbf{X}) \tag{9}$$

As in the derivation of (8) we obtain, by applying the invariance principle to (9)

$$\begin{aligned} &\Phi_1(t, t_0, t_1, t_2, \ldots t_N, \mathbf{F}^t(\mathbf{X}), \mathbf{X}) = \\ &\Phi_1(t-t_0, 0, t_1-t_0, t_2-t_0, \ldots t_N-t_0, \mathbf{F}^t(\mathbf{X},)\, \mathbf{X}) = \\ &\Phi_2(t-t_0, t_1-t_0, t_2-t_0, \ldots t_N-t_0, \mathbf{F}^t(\mathbf{X}), \mathbf{X}) = \\ &\Phi(t-t_0(\mathbf{X}), \mathbf{F}^t(\mathbf{X}), \mathbf{X}) \end{aligned} \tag{10}$$

where the last two equalities are a simple definition of Φ_2 and Φ while the dependence of the reaction on $t_1-t_0, t_2-t_0, \ldots t_N-t_0$ is taken into account by the last argument in Φ. Therefore, by using the princple of invariance we have obtained a general defining equation for an inhomogeneously ageing material

$$\sigma(t, \mathbf{X}) = \Phi(t-t_0(\mathbf{X}), \mathbf{F}^t(\mathbf{X}), \mathbf{X}) \tag{11}$$

In the special case when Φ is independent of the third argument $\mathbf{X}$, the equation obtained has a simple meaning: in local time $t' = t - t_0(\mathbf{X})$ the properties of the material elements are described by the identical relationship for all $\mathbf{X}$

$$\sigma'(t', \mathbf{X}) = \Phi(t', \mathbf{F}^{*t'}(\mathbf{X})) \tag{12}$$

The local time principle formulated in such a manner can underlie the derivation of known defining equations of inhomogeneously ageing media /8/.

In constructing specific constitutive equations of media with a memory the Boltzmann superposition principle often turns out to be effective, and its general formulation can be given as follows: the consequence $u(t)$ caused by several increments of the cause $v(\tau)$ for $\tau \leqslant t$ is the sum of the effects caused by each of the increments of the cause separately. Taking account of the invariance condition for the mathematical formulation of the Boltzmann principle, we introduce the function $l_1(t-t_0, s)$ that equals the quantity $-u(t)$ caused by applying a single step function v at the time $\tau = t - s$. We then have

$$u(t) = L^\circ v(t) = \int_0^{+\infty} l_1(t-t_0, s)\, d_s v^t(s) \equiv \tag{13}$$
$$\int_{-\infty}^{t} l(t-t_0, \tau-t_0)\, dv(\tau), \quad l(t, \tau) = -l_1(t, t-\tau)$$

The defining equations of media with a memory can be obtained from the laws for the behaviour of an elastic body by replacing the constants by Volterra operators of the form (13). In the case of finite strains, conversion of the elastic law to one of the infinite set of presented defining equation modes that contain no constraints on the mapping therein /1/ should precede the replacement mentioned. For example, by using the neo-Hooke equation

$$\sigma = -p\mathbf{I} + \mu(\mathbf{B} - \mathbf{I}\,\mathrm{tr}\,\mathbf{B}/3), \quad \mathbf{B} = \mathbf{F}\mathbf{F}^T$$

and the referred mode

$$\sigma = -p\mathbf{I} + \mathbf{F}(t)\,\Phi(\mathbf{C}^t)\,\mathbf{F}^T(t)$$

we obtain the equation

$$\sigma = -p\mathbf{I} + \mathbf{F}(t)\{\mu^\circ(\mathbf{I} - \mathbf{C}^{-1}\,\mathrm{tr}\,\mathbf{C}\,3)(t)\}\,\mathbf{F}^T(t)$$

proposed in /9/ for certain polymers for a time-indpendent operator μ° of the form (13).

Therefore, unlike infinitesimal strains, the construction of a viscoelastic analogue of the elastic law is not unique for finite strains.

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A METHOD FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS USING DIFFERENTIABLE TRIGONOMETRIC FOURIER SERIES*

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A method for representing a function of two variables $u(x, y)$, that is defined in the square $\sigma = [0, \pi] \times [0, \pi]$, is presented in the form of a combination of polynomials and differentiable trigonometric series. Such a representation enables problems to be solved in which the unknown function is defined from partial differential equations and has some partial derivatives at the border of the square domain of higher order than the order of the equation. Expansion in a trigonometric series is carried out by a system of functions $\{\sin mx, m = 1, 2, 3 \ldots\}$ that is full in $[0, \pi]$ and in a double series by a system of functions $\{\sin mx \sin ny, m, n = 1, 2, 3, \ldots\}$ that is full in σ. For solving real problems, expansion by such a system of functions can be preferable to expansion by an ordinary trigonometric system of sines and cosines /1, 2/. Using the representation of a function of two variables referred to above the problem of the bending of an anisotropic plate with non-uniform boundary conditions is solved.

1. Formulation and foundation of the method. *Definition 1.* A function $f(x), x \in [0, \pi]$ is even (odd) over $[0, \pi]$ relative to the point $\pi/2$ if $f(x) = f(\pi - x), x \in [0, \pi]$, $(f(x) = -f(\pi - x), x \in [0, \pi])$. A function $F(x, y), (x, y) \in \sigma$ is even or odd in x and y if the corresponding relations on the argument x or y are satisfied.

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